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1995 J. Phys. A: Math. Gen. 28 7181

(http://iopscience.iop.org/0305-4470/28/24/016)

J. Phys. A: Math. Gen. **28** (1995) 7181–7196. Printed in the UK

Gauge transformations for higher-order Lagrangians

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Received 1 August 1995

Abstract. Noether's symmetry transformations for higher-order Lagrangians are studied. A characterization of these transformations is presented, which is useful for finding gauge transformations for higher-order singular Lagrangians. The case of second-order Lagrangians is studied in detail. Some examples that illustrate our results are given; in particular, for the Lagrangian of a relativistic particle with curvature, Lagrangian gauge transformations are obtained, though there are not Hamiltonian gauge generators for them.

1. Introduction

Among the symmetries of a classical dynamical system described through an action principle, Noether's symmetries [3, 18, 23, 24] (i.e. those that leave the action invariant, up to boundary terms) play a central role. They are the usual symmetries considered in systems of physical interest, their characterization is very simple, and, most importantly, they are the kind of symmetries that we must consider when dealing with quantum systems; this is clear from the path-integral formulation, where the main ingredient is the classical action together with the measure in the space of field configurations.

Here we will consider continuous symmetries, either rigid or gauge. In the latter case, the infinitesimal transformation will depend upon arbitrary functions of time—in mechanics—or spacetime—in field theory. In order for these gauge transformations to exist the Lagrangian must be singular. In a first-order Lagrangian this means that the Hessian matrix with respect to the velocities is singular; it is so with respect to the highest derivatives in a higher-order case. Constants of motion appear associated to rigid symmetries whereas first-class Hamiltonian constraints appear associated to gauge symmetries [8]; in this case Noether's identities also appear.

For regular Lagrangians the constant of motion associated with a Noether's symmetry is, in fact, the generator of the symmetry when expressed in Hamiltonian formalism. For singular Lagrangians this statement is not always true: a Lagrangian Noether's transformation may not be projectable to phase space.

In [1, 3] several aspects of Noether's symmetries for first-order Lagrangians have been studied; in particular, the projectability of these transformations from Lagrangian to Hamiltonian formalism. Let us explain this point. Let $L(q, \dot{q})$ be a first-order

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Lagrangian and $\mathcal{F}L$ its associated Legendre transformation mapping velocity space to phase space: $\mathcal{F}L(q, \dot{q}) = (q, \hat{p})$, where $\hat{p}(q, \dot{q}) = \partial L/\partial \dot{q}$ are the momenta. Given a Noether's symmetry $\delta q(t, q, \dot{q})$ of L, the corresponding constant of motion $G_{\mathrm{L}}(t, q, \dot{q})$ turns out to be projectable to a function $G_{\mathrm{H}}(t, q, p)$ in phase space. This means that there is a function G_{H} whose pull-back $\mathcal{F}L^*(G_{\mathrm{H}})$ through the Legendre transformation is G_{L}; in other words, $G_{\mathrm{H}}(t, q, \hat{p}) = G_{\mathrm{L}}(t, q, \dot{q})$. (Notice that, for a singular Lagrangian, not every function in velocity space is projectable to a function in phase space, due to the singularity of the Legendre transformation.) Then there is a simple characterization of the functions G_{H} that correspond to a Noether's symmetry [13]. Finally, the function G_{H} acts as a kind of generator for the Noether's symmetry δq. If the functions $\delta q(t, q, \dot{q})$ are projectable to phase space, then G_{H} can be chosen (between the functions whose pull-back to velocity space is G_{L}) such that it generates the symmetry in the same way as for regular Lagrangians, i.e. through Poisson bracket; otherwise, G_{H} still generates the Noether's symmetry though not in such a simple way.

In this paper we extend these results to higher-order Lagrangians. For these Lagrangians there exists a Hamiltonian formulation, due to Ostrogradskiĭ; in the case of singular Lagrangians, Dirac's theory may be applied, and, for instance, the search for generators of symmetry transformations [11] is performed as for the first-order case. As we will see, when we look for Noether's symmetry transformations of a higher-order singular Lagrangian the situation is rather different from the first-order case. The most remarkable difference is that in the higher-order case the constant of motion G_{L} is not necessarily projectable to a function G_{H} in phase space.

To perform this analysis we make use of the results of [2, 5]. As it will be summarized in section 2, given a kth-order Lagrangian there are $k-1$ intermediate spaces $P_0 \xrightarrow{\mathcal{F}_0} P_1 \to \ldots P_{k-1} \xrightarrow{\mathcal{F}_{k-1}} P_k$ between those of Lagrangian (P_0) and Hamiltonian (P_k) formalisms, where $\mathcal{F}_0, \ldots, \mathcal{F}_{k-1}$ are the 'partial Legendre–Ostrogradskiĭ's transformations'. So the study of the projectability of a Lagrangian quantity (in P_0) to phase space (P_k) is more involved. In particular, unlike the first-order case, the constant of motion of a Noether's symmetry, although being projectable to the intermediate space P_1, is not necessarily projectable to the phase space.

Our characterization of Noether's transformations is especially relevant when looking for gauge transformations. For instance, in [13] there is a Lagrangian not possessing Hamiltonian gauge generators, but such that our method provides Lagrangian gauge symmetries for it. Another example is given by the Lagrangian of a conformal particle [16]: it has a Hamiltonian gauge symmetry that cannot be written in a covariant form despite the covariance of the Hamiltonian constraints; in this case our method allows us to construct a covariant Lagrangian gauge symmetry. In this paper a similar behaviour is shown to occur in a second-order Lagrangian, namely the curvature of the world-line of a relativistic particle: it will be shown that it has no Hamiltonian gauge generators, but two independent Lagrangian gauge transformations will be obtained for it.

The paper is organized as follows. In section 2 some results on higher-order Lagrangians are summarized. In section 3 Noether's transformations for higher-order Lagrangians are studied, and a characterization of them is introduced. In section 4 the case of second-order Lagrangians is developed in full detail. As an application of these results, in section 5 the example of the particle with the curvature as a Lagrangian is studied; other examples are also studied in the next section. The paper ends with a section with conclusions and an appendix about Hamiltonian symmetry transformations.

2. Higher-order Lagrangians

Here we present some results and notation from [15]. See also [2, 7, 20, 29] for higher-order Lagrangians and higher-order tangent bundles.

Let Q be an n-dimensional differentiable manifold with coordinates† $q = q_0$. On its higher-order tangent bundles $T^r Q$ we consider natural coordinates $(q_0, \ldots, q_r)$. A kth-order Lagrangian is a function $L: T^k Q \to \mathbf{R}$.

The Ostrogradskiĭ momenta are

$$\hat{p}^i = \sum_{j=0}^{k-i-1} (-1)^j \mathrm{D}_t^j \left(\frac{\partial L}{\partial q_{i+j+1}} \right) \qquad 0 \leqslant i \leqslant k-1 \tag{2.1}$$

where $\mathrm{D}_t = \partial/\partial t + \sum_i q_{i+1} \partial/\partial q_i$ is the total time derivative. Equivalently,

$$\hat{p}^{k-1} = \frac{\partial L}{\partial q_k} \qquad \hat{p}^{i-1} = \frac{\partial L}{\partial q_i} - \mathrm{D}_t \hat{p}^i . \tag{2.2}$$

Notice that $\hat{p}^i$ depends only on $q_0, \ldots, q_{2k-1-i}$.

In coordinates the Euler–Lagrange equations can be written $[L]_{q(t)} = 0$, with

$$\begin{aligned} [L] &= \sum_{r=0}^{k} (-1)^r \mathrm{D}_t^r \left(\frac{\partial L}{\partial q_r} \right) \\ &= \frac{\partial L}{\partial q_0} - \mathrm{D}_t \hat{p}^0 && (2.3) \\ &= \alpha - (-1)^{k-1} q_{2k} W && (2.4) \end{aligned}$$

where

$$\alpha = \frac{\partial L}{\partial q_0} - q_1 \frac{\partial \hat{p}^0}{\partial q_0} - \cdots - q_{2k-1} \frac{\partial \hat{p}^0}{\partial q_{2k-2}}$$

and W is the Hessian matrix with respect to the highest-order velocities,

$$W = \frac{\partial^2 L}{\partial q_k \, \partial q_k} .$$

Introducing the momenta step-by-step, for $0 \leqslant r \leqslant k$ an intermediate space P_r can be defined, with coordinates $(q_0, \ldots, q_{2k-1-r}; p^0, \ldots, p^{r-1})$. In particular, the Lagrangian and Hamiltonian spaces are $P_0 = T^{2k-1}Q$ and $P_k = T^*(T^{k-1}Q)$. Observe that P_k has a canonical Poisson bracket, for which $\{q_r^i, p_j^s\} = \delta_r^s \delta_j^i$.

The partial Ostrogradskiĭ's transformations $\mathcal{F}_r: P_r \to P_{r+1}$ can be introduced, with local expression

$$\mathcal{F}_r(q_0, \ldots, q_{2k-1-r}; p^0, \ldots, p^{r-1}) = (q_0, \ldots, q_{2k-2-r}; p^0, \ldots, p^{r-1}, \hat{p}^r) . \tag{2.5}$$

The 'total' Legendre–Ostrogradskiĭ's transformation is $\mathcal{F}L = \mathcal{F}_{k-1} \circ \ldots \circ \mathcal{F}_0: P_0 \to P_k$.

On P_r there exists an unambiguous evolution operator K_r, which is a vector field along $\mathcal{F}_r$, $K_r: P_r \to T(P_{r+1})$, satisfying certain conditions [15, theorem 4]. In coordinates it reads

$$\begin{aligned} K_r = q_1 \frac{\partial}{\partial q_0} + \cdots + q_{2k-1-r} \frac{\partial}{\partial q_{2k-2-r}} \\ + \left(\frac{\partial L}{\partial q_0} \right) \frac{\partial}{\partial p^0} + \left(\frac{\partial L}{\partial q_1} - p^0 \right) \frac{\partial}{\partial p^1} + \cdots + \left(\frac{\partial L}{\partial q_r} - p^{r-1} \right) \frac{\partial}{\partial p^r} . \end{aligned} \tag{2.6}$$

† Indices of coordinates are usually suppressed.

The various evolution operators are connected by

$$K_{r-1} \cdot \mathcal{F}_r^*(g) = \mathcal{F}_{r-1}^*(K_r \cdot g) \tag{2.7}$$

for $1 \leqslant r \leqslant k-1$; here $\mathcal{F}_r^*(g)$ denotes the pull-back of g through $\mathcal{F}_r$.

These intermediate evolution operators act as differential operators from functions in P_{r+1} to functions in P_r. They can be extended to act on time-dependent functions; for instance, given a time-dependent function in P_1, $g(t, q_0, \ldots, q_{2k-2}, p^0)$,

$$K_0 \cdot g = \mathcal{F}_0^* \left(\frac{\partial g}{\partial t}\right) + q_1 \mathcal{F}_0^* \left(\frac{\partial g}{\partial q_0}\right) + \cdots + q_{2k-1} \mathcal{F}_0^* \left(\frac{\partial g}{\partial q_{2k-2}}\right) + \frac{\partial L}{\partial q_0} \mathcal{F}_0^* \left(\frac{\partial g}{\partial p^0}\right).$$

By computing $K_0 \cdot g - \mathrm{D}_t \mathcal{F}_0^*(g)$ using the chain rule an interesting relation is obtained:

$$K_0 \cdot g = [L] \mathcal{F}_0^* \left(\frac{\partial g}{\partial p^0}\right) + \mathrm{D}_t \mathcal{F}_0^*(g)\,. \tag{2.8}$$

We assume that W has constant rank $n-m$. Then the $\mathcal{F}_r$ have constant rank $2kn - m$, since

$$\frac{\partial \hat{p}^r}{\partial q_{2k-1-r}} = (-1)^{k-1-r} W$$

and $P_{r+1}^{(1)} := \mathcal{F}_r(P_r)$ is then assumed to be a closed submanifold of P_{r+1} locally defined by m independent primary constraints ϕ_{r+1}^μ. The primary Hamiltonian constraints—those defining $P_k^{(1)}$—can be chosen to be independent of $p^0, \ldots, p^{k-2}$. Then the primary constraints of P_r can be obtained by applying K_r to the primary constraints of P_{r+1} [15, proposition 9]:

$$\phi_r^\mu := K_r \cdot \phi_{r+1}^\mu\,. \tag{2.9}$$

This is also true for $r = 0$. Indeed one can write evolution equations on each space P_r ($0 \leqslant r \leqslant k-1$); these equations are equivalent to the Euler–Lagrange equations. The first consistency conditions for these equations are just the constraints ϕ_r^μ defined above.

The primary constraints yield a basis for $\operatorname{Ker} W$:

$$\gamma_\mu = \mathcal{F}_{k-1}^* \left(\frac{\partial \phi_k^\mu}{\partial p^{k-1}}\right)$$

which can also be written as $(-1)^{k-r} \mathcal{F}_{r-1}^* \left(\partial \phi_r^\mu / \partial p^{r-1}\right)$, provided that the ϕ_r^μ are defined by (2.9). Notice that γ_μ depends only on $(q_0, \ldots, q_k)$. Then, a basis for $\operatorname{Ker} T(\mathcal{F}_r)$ is constituted by the vector fields

$$\Gamma_\mu^r = \gamma_\mu \frac{\partial}{\partial q_{2k-1-r}}\,.$$

These can be used to test the projectability of a function in P_r to P_{r+1}: $\Gamma_\mu^r \cdot g = 0$.

We notice also the commutation relations

$$\Gamma_\mu^r \cdot (K_r \cdot g) = \mathcal{F}_r^*(\Gamma_\mu^{r+1} \cdot g)$$

for $0 \leqslant r \leqslant k-1$, where Γ_μ^k is understood as $\Gamma_\mu^k \cdot g = \{g, \phi_k^\mu\}$.

Using the null vectors γ_μ, (2.8) and (2.9), we obtain, in particular, the primary Lagrangian constraints as

$$\phi_0^\mu = K_0 \cdot \phi_1^\mu = (-1)^{k-1} [L] \gamma_\mu = (-1)^{k-1} \alpha \gamma_\mu\,.$$

There is a Hamiltonian function in P_k, which is a projection of the Lagrangian energy function $E_0(q_0, \cdots, q_{2k-1}) = \hat{p}^0 q_1 + \ldots + \hat{p}^{k-1} q_k - L(q_0, \ldots, q_k)$; it can be chosen to be in the particular form

$$H = \sum_{r=0}^{k-2} p^r q_{r+1} + h(q_0, \ldots, q_{k-1}; p^{k-1})\,. \tag{2.10}$$

The usual presymplectic (Dirac's) analysis can be performed in $P_k^{(1)}$. In fact, there are stabilization algorithms for the dynamics of the intermediate spaces and all the constraints in P_r—not only the primary ones—are obtained applying K_r to all the constraints in P_{r+1} [15, theorem 8]. This result holds indeed at each step of the stabilization algorithms.

3. Noether's transformations

An infinitesimal Noether's symmetry [3, 18, 23, 24] (see also [4–6, 9, 19, 21]) is an infinitesimal transformation δq such that

$$\delta L = \mathrm{D}_t F$$

for a certain F. It yields a conserved quantity $G = \sum_{r=0}^{k-1} \hat{p}^r \delta q_r - F$, where $\delta q_r = \mathrm{D}_t^r \delta q$, since

$$[L]\delta q + \mathrm{D}_t G = 0$$

this is proved using the Euler–Lagrange equations (2.3) and the relation between the momenta.

So let us consider a $\delta q(t, q_0, \ldots, q_{2k-1})$, and a function $G_\mathrm{L}(t, q_0, \ldots, q_{2k-1})$ such that

$$[L]\delta q + \mathrm{D}_t G_\mathrm{L} = 0\,. \tag{3.1}$$

Notice that the highest derivative in this relation, q_{2k}, appears linearly, and its coefficient is

$$(-1)^k W \delta q - \frac{\partial G_\mathrm{L}}{\partial q_{2k-1}} = 0$$

so, contracting with the null vectors γ_μ we obtain that

$$\Gamma_\mu \cdot G_\mathrm{L} = 0\,.$$

That is to say, G_L is projectable to a function G_I in P_1,

$$G_\mathrm{L} = \mathcal{F}_0^*(G_\mathrm{I})\,.$$

Now, using (2.8), (3.1) becomes

$$[L]\left(\delta q - \mathcal{F}_0^*\left(\frac{\partial G_\mathrm{I}}{\partial p^0}\right)\right) + K_0 \cdot G_\mathrm{I} = 0\,.$$

Looking again at the coefficient of q_{2k} in this expression, we obtain

$$W\left(\delta q - \mathcal{F}_0^*\left(\frac{\partial G_\mathrm{I}}{\partial p^0}\right)\right) = 0$$

and so the parentheses enclose a null vector of W:

$$\delta q - \mathcal{F}_0^*\left(\frac{\partial G_\mathrm{I}}{\partial p^0}\right) = \sum_\mu r^\mu \gamma_\mu$$

for some $r^\mu(t, q_0, \ldots, q_{2k-1})$. Substituting this expression we obtain

$$K_0 \cdot G_\mathrm{I} + \sum_\mu r^\mu (\alpha \gamma_\mu) = 0\,. \tag{3.2}$$

So we have proved the following result:

Theorem 1. Let $\delta q(t, q_0, \ldots, q_{2k-1})$ be a Noether's transformation with conserved quantity G_L. Then G_L is projectable to a function G_I in P_1 such that†

$$K_0 \cdot G_I \underset{P_0^{(1)}}{\simeq} 0 \tag{3.3}$$

where $P_0^{(1)}$ is the primary Lagrangian constraint submanifold.

Conversely, given a function $G_I(t, q_0, \ldots, q_{2k-2}, p^0)$ satisfying this relation, if r^μ are functions such that $K_0 \cdot G_I = -\sum_\mu r^\mu(\alpha\gamma_\mu)$ then

$$\delta q = \mathcal{F}_0^* \left(\frac{\partial G_I}{\partial p^0}\right) + \sum_\mu r^\mu \gamma_\mu \tag{3.4}$$

is a Noether's transformation with conserved quantity $G_L = \mathcal{F}_0^*(G_I)$.

Notice that δq is not necessarily projectable to P_1, not to mention to phase space P_k; in fact, the projectability of δq is equivalent to the projectability of the functions r^μ.

There is also a certain indetermination in the functions r^μ [14]. For instance, if there are at least two primary Lagrangian constraints then one can add convenient combinations of these constraints to the r^μ, namely, an antisymmetric combination of the primary Lagrangian constraints, in a way that (3.2) is still satisfied; however, this change corresponds to adding a trivial gauge transformation [17] to the original transformation, and so we still have the same transformation on-shell (i.e. for solutions of the equations of motion). Another interesting case occurs when the primary Lagrangian constraints are not independent; in [14] the relation between this fact and Noether's transformations with vanishing conserved quantity is studied. For instance, one of the primary Lagrangian constraints, say $\chi = K_0 \cdot \psi$, may be identically vanishing, and so for $G_I = 0$ *any* value for the corresponding r is admissible to fulfil $K_0 \cdot G_I + r\chi = 0$. This yields a Noether's transformation

$$\delta q = r\, \mathcal{F}_0^* \left(\frac{\partial \psi}{\partial p^0}\right).$$

For instance, r might be an arbitrary function of time, thus yielding a gauge transformation. Summing up: unlike the case of a regular Lagrangian, where there is a one-to-one correspondence between Noether's transformations and conserved quantities, for a singular Lagrangian in general there is a whole family of Noether's transformations associated with a single conserved quantity.

4. Projectability of Noether's transformations in the case of second-order Lagrangians

In the first-order case, $k = 1$, the results of the previous section tell us that G_L is projectable to the phase space $P_1 = T^*Q$. As we will see shortly this is not true for a higher-order case $k \geqslant 2$. This means that there is no guarantee that we can write the conserved quantity in canonical variables, let alone to get the Noether's transformation in phase space: as we can read off from (3.4), this is not always possible even for the first-order case.

In order to clarify both issues, projectability of G_L and projectability of δq, which in fact we will see are related, we will perform a thorough study of the case $k = 2$, which will already show the basic features of the general picture for any k.

† $f \underset{M}{\simeq} 0$ means $f = 0$ on M (Dirac's weak equality).

Let us consider from now on the case $k = 2$. A basis for $\mathrm{Ker}\, T(\mathcal{F}L)$ is given [2] by the vector fields

$$\Gamma^0_{\mu_1} = \gamma_{\mu_1}\frac{\partial}{\partial q_3} \qquad \tilde{\Gamma}^0_{\mu'_1} = \gamma_{\mu'_1}\frac{\partial}{\partial q_2} + \eta_{\mu'_1}\frac{\partial}{\partial q_3}\,.$$

The index μ'_1 is a part of the indices μ_1 that corresponds to the splitting of the primary Hamiltonian constraints $\phi_2^{\mu_1}$ into the first-class ones, $\phi_2^{\mu'_1}$, and the second-class ones $\phi_2^{\mu''_1}$. The function $\eta_{\mu'_1}$ can be written as $\eta_{\mu'_1} = \partial\phi_2^{\mu_2}/\partial p^1$, where $\phi_2^{\mu_2} = \{\phi_2^{\mu'_1}, H\}$ are the secondary constraints in phase space (here μ'_1 and μ_2 run over the same set of indices, but are distinguished in order to label primary or secondary constraints).

It is easy to prove that the vector fields $\tilde{\Gamma}^0_{\mu_1}$ are projectable to the intermediate space P_1. In fact, since the definition of $\mathrm{Ker}\, T(\mathcal{F}L)$ requires that $\tilde{\Gamma}^0_{\mu'_1}(\mathcal{F}_0^*(p^0)) = 0$, we get immediately $\tilde{\Gamma}^0_{\mu'_1} \circ \mathcal{F}_0^* = \mathcal{F}_0^* \circ \Gamma^1_{\mu'_1}$ (as operators on functions of the intermediate space).

Now we can check the condition of projectability of G_{L} to P_2. Since $\Gamma^0_{\mu_1} \cdot G_{\mathrm{L}} = 0$, we only have to check whether $\tilde{\Gamma}^0_{\mu'_1} \cdot G_{\mathrm{L}}$ vanishes:

$$\begin{aligned}\tilde{\Gamma}^0_{\mu'_1} \cdot G_{\mathrm{L}} &= \tilde{\Gamma}^0_{\mu'_1} \cdot \mathcal{F}_0^*(G_{\mathrm{I}}) = \mathcal{F}_0^*(\Gamma^1_{\mu'_1} \cdot G_{\mathrm{I}}) = \Gamma^0_{\mu'_1} \cdot (K_0 \cdot G_{\mathrm{I}}) \\ &= \Gamma^0_{\mu'_1} \cdot (-r^\mu(\alpha\gamma_\mu)) = -(\Gamma^0_{\mu'_1} \cdot r^\mu)(\alpha\gamma_\mu) \\ &= -\alpha((\Gamma^0_{\mu'_1} \cdot r^\mu)\gamma_\mu) = -\alpha(\Gamma^0_{\mu'_1} \cdot (r^\mu\gamma_\mu)) = -\alpha(\Gamma^0_{\mu'_1} \cdot \delta q)\,. \end{aligned} \tag{4.1}$$

Notice that the projectability of G_{L} to P_1 depends on δq. In this argument we have used several commutation properties of the Γ's, but there are two details to point out.

First, $\Gamma^0_{\mu'_1} \cdot (\alpha\gamma_\mu) = 0$; this is a consequence of a more general result:

$$-\Gamma^0_\nu \cdot (\alpha\gamma_\mu) = \mathcal{F}L^*\{\phi_2^\mu, \phi_2^\nu\}$$

whose proof is immediate:

$$-\Gamma^0_\nu \cdot (\alpha\gamma_\mu) = \Gamma^0_\nu \cdot (K_0 \cdot \phi_1^\nu) = \mathcal{F}_0^*(\Gamma^1_\nu \cdot (K_1 \cdot \phi_2^\nu)) = \mathcal{F}_0^*(\mathcal{F}_1^*(\Gamma^2_\nu \cdot \phi_2^\nu)) = \mathcal{F}_0^*(\mathcal{F}_1^*\{\phi_2^\mu, \phi_2^\nu\})\,.$$

In particular, taking one of the constraints to be first class, $\{\phi_2^\mu, \phi_2^{\nu'}\} \underset{P_2^{(1)}}{\simeq} 0$, the result is zero.

Second, $(\Gamma^0_{\mu'_1} \cdot r^\mu)\gamma_\mu = \Gamma^0_{\mu'_1} \cdot (r^\mu\gamma_\mu)$, which is trivially true since the vector functions γ_μ are projectable.

Therefore we have obtained an expression for $\tilde{\Gamma}^0_{\mu'_1} \cdot G_{\mathrm{L}}$, and in general it can be different from zero. Notice that a sufficient condition for the projectability of G_{L} to P_2 is that δq be projectable to P_1. Notice also that the quantity $\alpha(\Gamma^0_{\mu'_1} \cdot \delta q)$ is insensitive to the indetermination of the functions r^μ which is mentioned at the end of the previous section.

Now we are going to consider that the conditions are met for the projectability of G_{L} to a function G_{H} in P_2, $\mathcal{F}L^*(G_{\mathrm{H}}) = G_{\mathrm{L}}$. The function G_{H} has a certain degree of arbitrariness because we can add to it arbitrary combinations of the primary as well as the secondary constraints in P_2. Let us extract consequences from our assumption. The function $\mathcal{F}_1^*(G_{\mathrm{H}})$ is one of the possible functions G_{I} considered in the previous section and therefore we can apply to it the results already obtained there. In particular,

$$K_0 \cdot (\mathcal{F}_1^*(G_{\mathrm{H}})) \underset{P_0^{(1)}}{\simeq} 0\,.$$

But since $K_0 \circ \mathcal{F}_1^* = \mathcal{F}_0^* \circ K_1$,

$$\mathcal{F}_0^*(K_1 \cdot G_{\mathrm{H}}) \underset{P_0^{(1)}}{\simeq} 0$$

which means that

$$K_1 \cdot G_H = \sum_{\mu_1} u_1^{\mu_1} \phi_1^{\mu_1} + \sum_{\mu_2} v_1^{\mu_2} \phi_1^{\mu_2} .$$

Here $\phi_1^{\mu_1}$ and $\phi_1^{\mu_2}$ are respectively the primary and the secondary constraints is P_1 (remember that μ_2 runs over the same indices as μ_1'). Notice that $\mathcal{F}_0^*(\phi_1^{\mu_1}) = 0$ and $\mathcal{F}_0^*(\phi_1^{\mu_2}) = (\alpha\gamma_{\mu_1'})$. Therefore,

$$K_0 \cdot \mathcal{F}_1^*(G_H) = \mathcal{F}_0^*(K_1 \cdot G_H) = \mathcal{F}_0^*(v_1^{\mu_1'})(\alpha\gamma_{\mu_1'})$$

and, according to the results of the previous section, the transformation

$$\delta q = \mathcal{F}_0^* \left(\frac{\partial \mathcal{F}_1^*(G_H)}{\partial p^0} \right) + \sum_{\mu_1'} \mathcal{F}_0^*(v_1^{\mu_1'}) \gamma_{\mu_1'} \tag{4.2}$$

is a Noether's transformation which is projectable to P_1.

If we define

$$G_I = \mathcal{F}_1^*(G_H) - \sum v_1^{\mu_1'} \phi_1^{\mu_1'}$$

since $\partial\phi_1^{\mu_1'}/\partial p^0 = \gamma_{\mu_1'}$, then $\delta q = \mathcal{F}_0^*\left(\partial G_I/\partial p^0\right)$ and

$$K_0 \cdot G_I = 0$$

where we have used $K_0 \phi_1^{\mu_1'} = -(\alpha\gamma_{\mu_1'})$.

Proposition 1. Let G_L be the conserved quantity of a Noether's transformation. The following statements are equivalent:

(i) G_L is projectable to a function G_H in P_2.
(ii) G_L is projectable to a function G_I in P_1 such that $K_0 \cdot G_I = 0$ (and then $\delta q = \mathcal{F}_0^*\left(\partial G_I/\partial p^0\right)$ is a Noether's transformation with conserved quantity G_L.)
(iii) Among the family of Noether's transformations whose conserved quantity is G_L, there is one transformation δq which is projectable to P_1.

The proof of the equivalence between the first and the second items is a direct consequence of the discussion preceding the proposition. Their equivalence to the third item follows also immediately from (4.1).

Now let us consider the case when δq is not only projectable to P_1 but also to P_2. This means that $v_1^{\mu_2}$ in (4.2) is projectable to P_2, $v_1^{\mu_2} = \mathcal{F}_1^*(v_2^{\mu_2})$. In such a case, taking into account that $K_1 \cdot \phi_2^{\mu_2} = \phi_1^{\mu_2}$, the function $G_H' := G_H - \sum v_2^{\mu_2} \phi_2^{\mu_2}$ satisfies

$$K_1 \cdot G_H' \underset{P_1^{(1)}}{\simeq} 0 \tag{4.3}$$

and δq can be expressed as

$$\delta q = \mathcal{F}L^* \left(\frac{\partial G_H'}{\partial p^0} \right)$$

which explicitly shows the projectability of δq to P_2.

There is still another way to write (4.3). If we define $K_E = \mathcal{F}_0^* \circ K_1 = K_0 \circ \mathcal{F}_1^*$, then (4.3) can be rewritten as

$$K_E \cdot G_H' = 0 .$$

The definition of K_E allows us to rewrite it as

$$K_E \cdot g = [L]\, \mathcal{F}L^* \left(\frac{\partial g}{\partial p^0} \right) + D_t\, \mathcal{F}L^*(g) .$$

This makes obvious in a direct way that $\delta q = \mathcal{F}L^*\left(\partial G'_{\rm H}/\partial p^0\right)$ is a Noether's transformation.

At this point we have the following result:

Proposition 2. Let $G_{\rm L}$ be the conserved quantity of a Noether's transformation. The following statements are equivalent:

(i) $G_{\rm L}$ is projectable to a function $G_{\rm H}$ in P_2 such that $K_E \cdot G_{\rm H} = 0$ (or equivalently $K_1 \cdot G_{\rm H} \underset{P_1^{(1)}}{\simeq} 0$) (and then $\delta q = \mathcal{F}L^*\left(\partial G_{\rm H}/\partial p^0\right)$ is a Noether's transformation with conserved quantity $G_{\rm L}$.)

(ii) Among the family of Noether's transformations whose conserved quantity is $G_{\rm L}$, there is one transformation δq which is projectable to P_2.

Now there is a subtle point. Is there a Hamiltonian symmetry $\delta_{\rm H}$ such that $\delta_{\rm H} q = \partial G_{\rm H}/\partial p^0$? As it is explained in the appendix, this is true only when (A.2) is also satisfied, and so we have the following result:

Proposition 3. Let $G_{\rm H}$ be a function in P_2. The following statements are equivalent:

(i) $K_1 \cdot G_{\rm H} = 0$.

(ii) $G_{\rm H}$ is the generator of a Hamiltonian symmetry transformation such that $\delta q = \mathcal{F}L^*(\delta_{\rm H} q)$, where $\delta_{\rm H} q = \{q_0, H\}$, is a Noether's transformation with conserved quantity $\mathcal{F}L^*(G_{\rm H})$.

This result can be directly generalized to any Lagrangian of order $k \geqslant 2$: the condition for a function $G_{\rm H}$ in P_k to be a generator of a Noether's Hamiltonian symmetry is

$$K_{k-1} \cdot G_{\rm H} = 0. \tag{4.4}$$

To summarize this section, we have started with a general Lagrangian Noether's transformation and we have examined some conditions to be satisfied by it, each one more restrictive, the latter being that of a Noether's Hamiltonian symmetry transformation. Therefore a conserved quantity of a Noether's transformation lies in one of the four different cases depicted by the previous propositions.

5. Application to the particle with curvature

Given a path $x(t)$ in Minkowski space $\mathbf{R}^d$, we write x_n for its nth time derivative, and e_n for the vectors obtained by orthogonalizing—if possible—the vectors $x_1, x_2, \ldots$. For instance,

$$e_1 = x_1 \tag{5.1a}$$

$$e_2 = x_2 - \frac{(x_2 e_1)}{(e_1 e_1)} e_1 \tag{5.1b}$$

$$e_3 = x_3 - \frac{(x_3 e_2)}{(e_2 e_2)} e_2 - \frac{(x_3 e_1)}{(e_1 e_1)} e_1 \,. \tag{5.1c}$$

We also write Δ_n for the Gramm determinant of the vectors $x_1 \ldots x_n$:

$$\Delta_n = \det((x_i x_j))_{1 \leqslant i,j \leqslant n} \,.$$

For a relativistic particle we consider a Lagrangian proportional to the curvature of its world line [2, 22, 25, 26],

$$L = \alpha \frac{\sqrt{\Delta_2}}{\Delta_1} = \alpha \frac{\sqrt{(x_1 x_1)(x_2 x_2) - (x_1 x_2)^2}}{(x_1 x_1)} \tag{5.2}$$

where α is a constant parameter.

Obviously e_1, e_2, e_3 are mutually orthogonal. Moreover,

$$(e_2x_2) = (e_2e_2) = \frac{\Delta_2}{\Delta_1} \qquad (e_2x_3) = \frac{\dot{\Delta}_2}{2\Delta_2}$$
$$(e_3x_3) = (e_3e_3) = \frac{\Delta_3}{\Delta_2}.$$

We shall also need

$$\dot{e_2} = e_3 + \left(\frac{\dot{\Delta}_2}{2\Delta_2} - \frac{\dot{\Delta}_1}{2\Delta_1}\right) e_2 - \frac{\Delta_2}{\Delta_1^2} e_1 .$$

The partial Ostrogradskiĭ's transformations are

$$\begin{array}{ccccc} P_0 = T^3(\mathbf{R}^d) & \xrightarrow{\mathcal{F}_0} & P_1 & \xrightarrow{\mathcal{F}_1} & P_2 = T^*(T(\mathbf{R}^d)) \\ (x_0, x_1, x_2, x_3) & \mapsto & (x_0, x_1, x_2, \hat{p}^0) & & \\ & & (x_0, x_1, x_2, p^0) & \mapsto & (x_0, x_1, p^0, \hat{p}^1) \end{array}$$

where the momenta are defined by

$$\hat{p}^1 := \frac{\partial L}{\partial x_2} = \frac{\alpha}{\sqrt{\Delta_2}} e_2 \tag{5.3}$$

$$\hat{p}^0 := \frac{\partial L}{\partial x_1} - \mathrm{D}_t \hat{p}^1 = -\frac{\alpha}{\sqrt{\Delta_2}} e_3 \tag{5.4}$$

for the last computation we have used

$$\frac{\partial L}{\partial x_1} = -\frac{\alpha}{\sqrt{\Delta_2}} \left(\frac{\dot{\Delta}_1}{2\Delta_1} e_2 + \frac{\Delta_2}{\Delta_1^2} e_1\right).$$

More precisely, P_0 is not all $T^3(\mathbf{R}^d)$, but the open subset defined by $\Delta_1 > 0$, $\Delta_2 > 0$. Then the vectors x_1 and x_2 are linearly independent, and so are e_1 and e_2. Similar remarks hold for P_1 and P_2.

The singularity of the partial Ostrogradskiĭ's transformations is due to the singularity of the Hessian matrix

$$W := \frac{\partial^2 L}{\partial x_2 \, \partial x_2} = -\frac{\partial \hat{p}^0}{\partial x_3} = \frac{\partial \hat{p}^1}{\partial x_2}.$$

In our case,

$$W_{\mu\nu} = \frac{\alpha}{\sqrt{\Delta_2}} \left(\eta_{\mu\nu} - \frac{e_{1\mu}}{\sqrt{(e_1e_1)}} \frac{e_{1\nu}}{\sqrt{(e_1e_1)}} - \frac{e_{2\mu}}{\sqrt{(e_2e_2)}} \frac{e_{2\nu}}{\sqrt{(e_2e_2)}}\right) \tag{5.5}$$

whose rank is $d-2$ in its domain.

The intermediate evolution operators are

$$\begin{aligned} K_1 : &= x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_1} + \frac{\partial L}{\partial x_0} \frac{\partial}{\partial p^0} + \left(\frac{\partial L}{\partial x_1} - p^0\right) \frac{\partial}{\partial p^1} \\ &= x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_1} - \left(p^0 + \frac{\alpha}{\sqrt{\Delta_2}} \left(\frac{\Delta_2}{\Delta_1^2} e_1 + \frac{\dot{\Delta}_1}{2\Delta_1} e_2\right)\right) \frac{\partial}{\partial p^1} \end{aligned} \tag{5.6}$$

$$\begin{aligned} K_0 : &= x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + \frac{\partial L}{\partial x_0} \frac{\partial}{\partial p^0} \\ &= x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} \end{aligned} \tag{5.7}$$

and the Euler–Lagrange equations (in P_0) are

$$[L] = \frac{\partial L}{\partial x_0} - \mathrm{D}_t \hat{p}^0 = 0 .$$

5.1. Constraints

The energy in P_1 is $E_1 := (p^0 x_1) + (\hat{p}^1 x_2) - L(x_0, x_1, x_2) = (p^0 x_1)$, so we take as a Hamiltonian

$$H = (p^0 x_1) . \tag{5.8}$$

Due to the rank of the Hessian matrix W, the definition of $\hat{p}^1$—the last partial Ostrogradskiĭ's transformation—introduces two constraints in the Hamiltonian space P_2. These constraints are obtained immediately from the relations satisfied by e_2, and we take them as

$$\phi_2^1 = (p^1 x_1) \tag{5.9a}$$

$$\psi_2^1 = \frac{1}{2}\left((p^1 p^1) - \frac{\alpha^2}{(x_1 x_1)}\right). \tag{5.9b}$$

We have

$$\{\phi_2^1, \psi_2^2\} = 2\psi_2^1 .$$

Proceeding with the Hamiltonian stabilization we obtain secondary constraints

$$\phi_2^2 = \{\phi_2^1, H\} = -(p^0 x_1) = -H \tag{5.10a}$$

$$\psi_2^2 = \{\psi_2^1, H\} = -(p^0 p^1) \tag{5.10b}$$

for which

$$\begin{pmatrix} \{\phi_2^1, \phi_2^2\} & \{\phi_2^1, \psi_2^2\} \\ \{\psi_2^1, \phi_2^2\} & \{\psi_2^1, \psi_2^2\} \end{pmatrix} = \begin{pmatrix} -\phi_2^2 & \psi_2^2 \\ -\psi_2^2 & \frac{\alpha^2}{(x_1 x_1)^2}\phi_2^2 \end{pmatrix}.$$

Finally, we obtain a tertiary constraint

$$\psi_2^3 = \{\psi_2^2, H\} = (p^0 p^0) \tag{5.11}$$

whereas $\{\phi_2^2, H\} = \{\psi_2^3, H\} = 0$. The Poisson bracket of ψ_2^3 with the primary constraints is zero.

Notice that all the constraints are first class, but the Poisson bracket between the two secondary constraints is the tertiary constraint:

$$\{\phi_2^2, \psi_2^2\} = \psi_2^3 .$$

The constraints in P_1 are obtained by applying the operator K_1 to the Hamiltonian constraints. We have

$$K_1 \cdot \phi_2^1 =: \phi_1^1 = -(p^0 x_1)$$

$$K_1 \cdot \psi_2^1 =: \psi_1^1 = -(p^0 \hat{p}^1) = -\frac{\alpha}{\sqrt{\Delta_2}}(p^0 e_2)$$

$$K_1 \cdot \phi_2^2 = \frac{(x_1 x_2)}{(x_1 x_1)}\phi_1^1 + \frac{\sqrt{\Delta_2}}{\alpha}\psi_1^1 \simeq 0$$

$$K_1 \cdot \psi_2^2 = (p^0 p^0) - \frac{(x_1 x_2)}{(x_1 x_1)}\psi_1^1 - \frac{\alpha\sqrt{\Delta_2}}{(x_1 x_1)^2}\phi_1^1 \simeq (p^0 p^0) =: \psi_1^2$$

$$K_1 \cdot \psi_2^3 = 0 .$$

Instead of defining $\psi_1^2 = K_1 \cdot \psi_2^2$ we prefer, for simplicity, to use $\psi_1^2 = (p^0 p^0)$, which defines the same constraint submanifold. With this convention, $\mathcal{F}_1^*(\phi_2^{i+1}) = \phi_1^i$, $\mathcal{F}_1^*(\psi_2^{i+1}) = \psi_1^i$.

Similarly from the intermediate constraints ϕ_1^1, ψ_1^1 and ψ_1^2 we obtain the Lagrangian constraints:

$$K_0 \cdot \phi_1^1 = 0 \qquad K_0 \cdot \psi_1^1 =: \psi_0^1 = (\hat{p}^0 \hat{p}^0) \qquad K_0 \cdot \psi_1^2 = 0 .$$

Again $\mathcal{F}_0^*(\phi_1^{i+1}) = \phi_0^i$ $\mathcal{F}_0^*(\psi_1^{i+1}) = \psi_0^i$.

From the expression of the Hessian matrix it is obvious that $\operatorname{Ker} W = \langle e_1, e_2 \rangle$. Indeed,

$$\gamma_\phi = \mathcal{F}_1^* \left(\frac{\partial \phi_2^1}{\partial p^1} \right) = -\mathcal{F}_0^* \left(\frac{\partial \phi_1^1}{\partial p^0} \right) = x_1$$

$$\gamma_\psi = \mathcal{F}_1^* \left(\frac{\partial \psi_2^1}{\partial p^1} \right) = -\mathcal{F}_0^* \left(\frac{\partial \psi_1^1}{\partial p^0} \right) = \hat{p}^1 .$$

5.2. Hamiltonian gauge transformations

We are going to show that the model does not have any Hamiltonian gauge transformation constructed from a generating function.

According to the appendix, we look for a generator of the form (A.3), and apply the algorithm (A.4). We first consider

$$G_0 = f\phi^1 + g\psi^1 \tag{5.12}$$

with f and g functions to be determined.

Then

$$G_1 = -f\phi^2 - g\psi^2 + f'\phi^1 + g'\psi^1 \tag{5.13}$$

for certain f', g'. We compute

$$\{\phi^1, G_1\} = (f - \{\phi^1, f\})\phi^2 - (g + \{\phi^1, g\})\psi^2 + \text{PFC}$$

$$\{\psi^1, G_1\} = -\left(\frac{\alpha^2}{(x_1 x_1)^2} g + \{\psi^1, f\} \right) \phi^2 + (f - \{\psi^1, g\})\psi^2 + \text{PFC}$$

and so to fulfil the test (A.4*c*.c) the expressions in parentheses must be weakly vanishing.

Now

$$G_2 = f\phi^3 + g\psi^3 + (\{f, H\} - f')\phi^2 + (\{g, H\} - g')\psi^2 + f''\phi^1 + g''\psi^1 \tag{5.14}$$

for some f'', g''. The test for G_2 requires to compute

$$\begin{aligned}\{\phi^1, G_2\} = \{\phi^1, g\}\psi^3 &+ \left(\{\phi^1, \{f, H\} - f'\} - \{f, H\} + f'\right)\phi^2 \\ &+ \left(\{\phi^1, \{g, H\} - g'\} + \{g, H\} - g'\right)\psi^2 + \text{PFC}\end{aligned}$$

$$\begin{aligned}\{\psi^1, G_2\} = \{\psi^1, g\}\psi^3 &+ \left(\{\psi^1, \{f, H\} - f'\} + \frac{\alpha^2}{(x_1 x_1)^2}(\{g, H\} - g') \right)\phi^2 \\ &+ \left(\{\psi^1, \{g, H\} - g'\} - \{f, H\} + f'\right)\psi^2 + \text{PFC} .\end{aligned}$$

In order that these expressions be strongly primary first-class constraints, the coefficients of ψ^3, ϕ^2 and ψ^2 must be weakly vanishing. From the coefficients of ψ^3 we obtain in particular that $\{\phi^1, g\}$ and $\{\psi^1, g\}$ are weakly vanishing. Looking at the coefficients of ψ^2 in the test for G_1 we obtain that f and g are weakly vanishing, so that the generator G is strongly vanishing: it becomes ineffective, since it leaves all solutions invariant.

5.3. Lagrangian gauge transformations

The model has two independent Noether's gauge transformations.

One of them is just the reparametrization. It arises easily from the fact that $\phi_0^1 := K_0 \cdot \phi_1^1 = 0$, i.e. one of the primary Lagrangian constraints is identically vanishing. This fact yields a Noether's transformation with vanishing conserved quantity, $G_L = 0$. According to the discussion on these transformations, we obtain a gauge transformation

$$\delta \boldsymbol{x} = \varepsilon(t) \boldsymbol{x}_1 \tag{5.15}$$

since $\gamma_\phi = \boldsymbol{x}_1$; this is just a reparametrization.

The other transformation comes from $G_I = \varepsilon(t)\psi_1^2 = \varepsilon(t)(\boldsymbol{p}^0\boldsymbol{p}^0)$, for which $G_L = \varepsilon(t)\psi_0^1 = \varepsilon(t)(\hat{\boldsymbol{p}}^0\hat{\boldsymbol{p}}^0)$. Then

$$K_0 \cdot G_I = \dot{\varepsilon}(t)\psi_0^1$$

so according to (3.2) we have $r = \dot{\varepsilon}(t)$, and since $\gamma_\psi = \hat{\boldsymbol{p}}^1$ we obtain

$$\delta \boldsymbol{x} = 2\varepsilon(t)\hat{\boldsymbol{p}}^0 + \dot{\varepsilon}(t)\hat{\boldsymbol{p}}^1 . \tag{5.16}$$

See [28] for a geometric interpretation of this transformation.

It can be shown that these transformations coincide with those obtained in [27] by considering a first-order Lagrangian when the supplementary variables are written in terms of derivatives of $\boldsymbol{x}$.

Notice that these transformations and their generating functions G_I are projectable to the Hamiltonian space; however, as we have explained at the end of the preceding section, they do not yield Hamiltonian gauge transformations, as it can be easily checked.

6. Other examples

Here we consider two simple examples of second-order singular Lagrangians to illustrate our procedure.

(i) $L(x_0, x_1, x_2) = x_2$.

The momenta are $\hat{p}^1 = 1$ and $\hat{p}^0 = 0$.

There are two Hamiltonian constraints, $\phi_2^1 = 1 - p^1$ and $\phi_2^2 = p^0$. In the intermediate space there is one constraint, $\phi_1^1 = p^0$. And finally there are no Lagrangian constraints.

Let us look for a gauge Noether's transformation 'generated' by a function $G_I = \varepsilon(t)p^0$. We obtain $K_0 \cdot G_I = 0$, so it satisfies the required condition, and the transformation is $\delta x = \mathcal{F}_0^*(\partial G_I/\partial p^0) = \varepsilon(t)$; this says that $x(t)$ is completely arbitrary, which, of course, is a consequence of the fact that $[L] = 0$ identically.

Notice that G_I projectable to a function $G_H = \dot{\varepsilon}(p^1 - 1) + \varepsilon p^0$ in the Hamiltonian space. For this function $K_1 \cdot G_H = 0$, and so in this case we obtain a Hamiltonian gauge transformation, $\delta x^0 = \varepsilon$, $\delta x^1 = \dot{\varepsilon}$, $\delta p^0 = \delta p^1 = 0$.

(ii) $L(x_0, x_1, x_2) = \frac{1}{2}(x_1 x_1)$.

This is a first-order Lagrangian, but let us treat it as a second-order one. The momenta are $\hat{p}^1 = 0$ and $\hat{p}^0 = x_1$.

In this example there are no Lagrangian constraints. In the intermediate space there is one constraint, $\phi_1^1 = p^0$. There are two Hamiltonian constraints, $\phi_2^1 = p^1$ and $\phi_2^2 = x_1 - p^0$; they are second class. In the intermediate space we have $\phi_1^1 = x_1 - p^0$ and $\phi_1^2 = x_2$. And finally we obtain two Lagrangian constraints, $\phi_0^1 = x_2$ and $\phi_0^2 = x_3$.

As usual let us look for a function $G_I = f x_2$. Now we find that $K_0 \cdot G_I = (K_0 \cdot f)x_2 + \mathcal{F}_0^*(f)x_3$; this is to vanish on the primary Lagrangian constraint submanifold, so necessarily we have $\mathcal{F}_0^*(f) \simeq 0$, $G_{\rm I} \cong 0$ and therefore there are no Noether's gauge transformations; this was expected since the solutions of the equations of motion are paths of constant velocity.

Now let us look for the rigid Noether's transformations of this Lagrangian. Due to the constraints of the intermediate space P_1, we try a function $G_{\rm I}(t, x_0, x_1)$. We obtain

$$K_0 \cdot G_{\rm I} = \frac{\partial G_{\rm I}}{\partial t} + x_1 \frac{\partial G_{\rm I}}{\partial x_0} + x_2 \frac{\partial G_{\rm I}}{\partial x_1}.$$

Since this has to vanish on the surface $x_2 \simeq 0$, we obtain the condition $\partial G_{\rm I}/\partial t + x_1 \partial G_{\rm I}/\partial x_0 = 0$, from which $G_{\rm I} = g(x_1 t - x_0, x_1)$; this yields two independent transformations, which are computed using the other term, the coefficient of x_2, $r = \partial G_{\rm I}/\partial x_1$.

7. Conclusions

In this paper we have studied Noether's symmetries for higher-order Lagrangians. This study is performed by using some intermediate spaces between those of Lagrangian and Hamiltonian spaces. We have seen that a conserved quantity of a Noether's transformation can be characterized in terms of a function in the first intermediate space satisfying a certain condition; this is also useful to find gauge transformations when the Lagrangian is singular.

The issue of projectability to the phase space of the Lagrangian conserved quantities as well as of the transformations themselves becomes quite a lot more involved than in the first-order case. To get a clearer picture of the subject we have made a thorough study of the second-order case, where the structures of the general higher-order case already show up. As a consequence of this study, we present a variety of cases covering all the possibilities with regard to the projectability (or partial projectability) of the quantities involved.

We give also some examples that illustrate several cases that appear in our analysis. In particular, the example of section 5 does not possess Hamiltonian gauge generators, in spite of the fact that it has Lagrangian Noether's transformations which are projectable to the Hamiltonian space.

Acknowledgments

We thank C Batlle for bringing to our attention the possible non-existence of Hamiltonian gauge generators for the model in section 5. We also thank J Roca for useful comments about this model.

Appendix. Gauge transformations in the Hamiltonian formalism

In this appendix we recall some results from [11]. We call *dynamical symmetry transformations* those transformations which map solutions of some equations of motion into solutions.

In the Dirac's Hamiltonian formalism, the *necessary and sufficient condition* for a function $G_{\rm H}(q, p; t)$ to generate, through Poisson bracket,

$$\delta f = \{f, G_{\rm H}\}$$

an infinitesimal dynamical symmetry transformation is that $G_{\rm H}$ be a *first-class function* and satisfy

$$\{G_{\rm H}, H\} + \frac{\partial G_{\rm H}}{\partial t} \underset{P^{(f)}}{\cong} \text{PFC} \tag{A.1a}$$

$$\{\text{PFC}, G_{\rm H}\} \underset{P^{(f)}}{\cong} \text{PFC} \tag{A.1b}$$

where $P^{(f)}$ is the submanifold defined by all the Hamiltonian constraints in phase space, PFC stands for any primary first-class Hamiltonian constraint, and the notation $f \underset{M}{\cong} 0$ means $f \underset{M}{\simeq} 0$ and $\mathrm{d}f \underset{M}{\simeq} 0$ (Dirac's strong equality).

This conditions can be equivalently expressed in a more compact form:

$$K \cdot G_{\rm H} \underset{V^{(f)}}{\cong} 0$$

where $V^{(f)}$ is the surface defined by all the Lagrangian constraints in velocity space and K is the time-evolution operator K for first-order Lagrangians—see, for instance, [12]. Though in [11] this is proved for first-order Lagrangians, it can be shown that this is also true for higher-order Lagrangians. More precisely, the condition is

$$K_{k-1} \cdot G_{\rm H} \underset{P_{k-1}^{(f)}}{\cong} 0 \tag{A.2}$$

where $P_{k-1}^{(f)}$ is the surface defined by all the constraints in the space P_{k-1}.

More particularly, we call a *gauge transformation* a dynamical symmetry transformation which depends on arbitrary functions of time. The general form for a generator of a Hamiltonian gauge transformation, depending on one arbitrary function, can be taken as

$$G_{\rm H}(q, p; t) = \sum_{k\geqslant 0} \epsilon^{(-k)}(t)\, G_k(q, p) \tag{A.3}$$

where $\epsilon^{(-k)}$ is a kth primitive of an arbitrary function of time ϵ.

To find a gauge generator, the characterization (A.2) or (A.1) of $G_{\rm H}$ as a dynamical symmetry generator splits yielding the following constructive algorithm, where strong equalities have been changed to normal equalities [11]:

$$G_0 = \text{PFC} \tag{A.4a}$$

$$\{G_k, H\} + G_{k+1} = \text{PFC} \tag{A.4b}$$

$$\{\text{PFC}, G_k\} \underset{P^{(f)}}{\cong} \text{PFC}\,. \tag{A.4c}$$

It is noticed, therefore, that though there may be second-class constraints, the generators of Hamiltonian gauge transformations are built up of *first-class constraints*, and, according to (A.4*a*), are headed by a primary one.

Some results on the existence of a basis of primary first-class Hamiltonian constraints each one yielding a gauge transformations are known: this is guaranteed under some regularity conditions [10], namely the constancy of the rank of Poisson brackets among constraints and the non-appearance of ineffective constraints. If these Hamiltonian gauge transformations exist, their pull-back constitutes a complete set of Lagrangian gauge transformations.

On the other hand, as we have said in the introduction, there are examples of first-order Lagrangians for which Hamiltonian gauge generators do not exist, whereas they have Lagrangian gauge transformations [13]. In this paper we have seen that this also happens for a relativistic particle with Lagrangian proportional to the curvature.

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